

Exact ground states for the four electron problem in a two-dimensional finite Hubbard square system.

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Abstract

We present exact explicit analytical results describing the exact ground state of four electrons in a two dimensional square Hubbard cluster containing 16 sites taken with periodic boundary conditions. The presented procedure, which works for arbitrary even particle number and lattice sites, is based on explicitly given symmetry adapted base vectors constructed in \mathbf{r} space. The Hamiltonian acting on these states generates a closed system of 85 linear equations providing by its minimum eigenvalue the exact ground state of the system. The presented results, described with the aim to generate further creative developments, not only show how the ground state can be exactly obtained and what kind of contributions enter in its construction, but emphasize further characteristics of the spectrum. On this line i) possible explications are found regarding why weak coupling expansions often provide a good approximation for the Hubbard model at intermediate couplings, or ii) explicitly given low lying energy states of the kinetic energy, avoiding double occupancy, suggest new roots for pairing mechanism attracting decrease in the kinetic energy, as emphasized by kinetic energy driven superconductivity theories.

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I. INTRODUCTION

The growing interest in systems of highly correlated electrons has proposed many theoretical descriptions where the dominant effects are treated in the frame of the Hubbard model (Hubbard 1963). Since the exact solution of the model is known only in one dimension (Lieb and Wu 1968), and the majority of driving forces in the field, as superconductivity in cuprates, are two dimensional (2D) problems, a great variety of approximate treatments have been proposed (Baeriswyl et al. 1995) in order to accommodate a suitable theoretical framework. Despite several years of intensive studies, it is apparent that the necessary theoretical skills and tools to deal with this problem are still in fact relatively poor (Lieb 1994).

While most of the ongoing effort relating the Hubbard model is being concentrated on the general case of large electron density, more than ten years ago it has been recognized that the less analysed low-density limit not only that retain main aspects related to the model behaviour (Falicov and Proetto 1993, Papavassiliou and Yartsev 1992), but can provide key knowledge which could drive further at least nonperturbative developments (Fabrizio, Parola and Tosatti 1991), firstly because is more hope to find non-approximated solutions holding by their nature essential information. The history of this background has started with the exact solution of the two-electron ground state problem on an arbitrary large torus (Chen and Mei 1980, Mei and Chen 1988), solution of the two electron problem in 2D (Parola et al. 1990), and continued with the three electron problem extended nonperturbatively in the low density limit (Fabrizio, Parola and Tosatti 1991). The numerical treatment of the four electron problem on four sites follows by the test of the Bardeen-Cooper-Schrieffer and resonant-valence-bond wave functions as approximated ground states of the Hubbard model (Falicov and Proetto 1993), test of the unrestricted Hartree-Fock solution (Louis et al. 1992), the calculation of Huckel-Hubbard correlation diagrams (Zhu and Jiang 1993), and deduction of absorption spectra of TCNQ particles (Papavassiliou and Yartsev 1992). The concurrently made numerical developments on 4×4 clusters around half filling (Galan and Verges 1991, Parola et al. 1991) corroborated by group-theoretical studies of the 4×4 square system (Fano, Ortolani and Parola 1992) also lead to the development of the numerical description of the 4 particle problem for energy level statistics (Bruus and d'Auriac 1996, 1997). In the last years the study of symmetry properties for Hubbard clusters in

general has continued (Tjemberg 1998), extensions to Hubbard-Holstein model for clusters holding 4 electrons has been given (Acquarone et al. 1999), and for spinless fermion case numerical characterization of the 4 particle problem goes up to clusters of 20×20 extension (Zhang and Henley 2004). Despite the invested efforts, exact analytical results holding essential information are not known at the moment in the problem.

The last period has provided strong new motivations for the further development of the study of the low concentration limit of the Hubbard model. The reason for this is that experimentally one started to encounter in condensed matter context rapidly increasing situations containing small number of particles confined in a system or device, as for example in the case of quantum dots (Maksym et al. 2000), quantum well structures (Kochereshko et al. 2003), mesoscopic systems (Halfpap 2001), experimental entanglement (Sackett et al. 2000) etc. Between the studied experiments, several are directly connected to the Hubbard model, as in the case of charge transfer complexes (Saito et al. 2001), quantum dots (Busser, Moreno and Dagotto 2004), or mesoscopic grains (Boyaci, Gedik and Kulik 2000). Furthermore, it has been observed that several measurements on strongly correlated systems are acceptable reproduced on small Hubbard clusters, as in the case of the X-ray absorption for nickelates studied on 4-site cluster (Okada 2004). The same can be stated for the 4×4 cluster case used in describing manganites (Helberg 2001, Srinitiwara Wong and Gehring 2002), photoemission intensities (Eroles, Batista and Aligia 1999), or thermodynamic properties provided by processes in the vicinity of the Fermi surface (Chiappe et al. 1999).

As can be observed, the accumulated knowledge to the present date regarding the extreme low density limit described by few particles present in the system, excepting the one and two particle cases, means only symmetry properties, and numerical results (often approximated), which usually conceal explicit properties which could generate creative advancement at the level of the theoretical description and deep understanding. This is a regrettable and unfortunate situation, since as is known from the accumulated experience in solving exactly many-body systems (Mattis 1993), or as several times has been accentuated stressed (Fabrizio, Parola and Tosatti 1991, Lieb 1994), key aspects of the unapproximated descriptions are often hidden in the few particle cases.

On the presented background, with the aim to fill up at least partially this gap for the two dimensional Hubbard case, and being driven by the intention to provide essential information for the subjects mentioned above, we present the exact ground state of four

interacting electrons in a 4×4 Hubbard cluster. Our main purpose is not to hide potential essential characteristics behind numerical results, but to present explicit expressions, basic resulting properties, and visible characteristics, which we strongly hope, are able to polarize the creative thinking, and advancement in the field.

Our results are based on an \mathbf{r} space description, which in our opinion is unjustly overshadowed in the last period in treating such problems, in contradiction with its deep ability to bring to light essential characteristics. We construct first a base vector set starting from symmetry properties, and then show how a closed system of linear equations containing 85 components can be constructed characterizing the ground state manifold and providing by its secular equation, through its minimum eigenvalue, the ground state wave function and ground state energy. After this step the properties of the ground state are analyzed. The concretely described situation is an $L \times L = N = 16$ square lattice with periodic boundary conditions in both directions, and $N_p = 4$ particles. But the method itself works for arbitrary even number of particles and arbitrary even L . The explicitly presented base vectors and equations providing the ground state, not only show how the ground state can be constructed and what kind of components build it up, but provides an insight into other properties of the spectrum as well. Especially the characteristics regarding the kinetic energy eigenstates should be mentioned on this line, important for understanding aspects related to perturbative expansions, or kinetic energy driven superconductivity.

The remaining part of the paper is structured as follows. Section II. presents the Hamiltonian, the deduction procedure and the ground state wave functions, Section III. describes physical properties of the deduced eigenstates, Sect. IV. presents the conclusions of the paper, while the Appendices A - B presenting mathematical details, close the presentation.

II. HAMILTONIAN AND GROUND STATE WAVE FUNCTIONS.

A. Presentation of the Hamiltonian

The Hamiltonian we use has the form of a standard Hubbard Hamiltonian

$$\hat{H} = -t \sum_{\langle i,j \rangle, \sigma} (\hat{c}_{i,\sigma}^\dagger \hat{c}_{j,\sigma} + H.c.) + U \sum_i \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow}, \quad (1)$$

where $\hat{c}_{i,\sigma}^\dagger$ creates an electron at site i with spin σ , t is the nearest-neighbour hopping amplitude, U is the on-site Coulomb repulsion, and $\langle i,j \rangle$ represents nearest-neighbour

sites, taken into account in the sum over sites only once. During our study we consider an $L \times L$ cluster, $N = L^2$, with periodic boundary conditions in both directions. All the presented results will relate \hat{H}/t , the only one microscopic parameter of the problem being $u = U/t$. The case $N = 16$ will be presented in details, although the method is applicable for arbitrary even N . Since $N_p = 4$, the band filling for the presented case is 0.125. We further mention that we consider during this paper $u > 0$.

B. The construction of the base wave vectors

1. The basic wave vector elements

In order to describe the Hilbert space region containing the ground state for the $N_p = 4$ particle problem, we use an \mathbf{r} -space representation for the wave vectors. In order to characterize this, first the lattice sites of the considered system are numbered starting from the down-left corner, the numbering being given along the lower lattice sites line, then going upward with the notation, as shown in Fig.1.a. In the considered system (taking into account that the ground state is a singlet state), three type of particle configurations may occur as depicted in Fig.1.b,c,d.

1) We can have two double occupancies at sites i and j represented by two dots at sites i and j (see Fig.1.b). 2) We may have a double occupancy at site i and two electrons with opposite spins at sites j and k . In this case, the double occupancy at site i it is denoted as before by a dot at i , while the two electrons with opposite spins placed on the different sites (j , and k) are depicted by a dotted line connecting the sites j and k (see Fig.1.c). Finally, we may have four single occupancies placed on different sites. From these, the electrons placed at i and j have spin σ , while the electrons at sites k and l have spin $-\sigma$. This situation will be represented by two continuous lines connecting the sites i and j , and k and l , respectively (see Fig.1.d). The notations i, j, k, l are representing the numbering of the sites as specified in Fig.1.a.

The mathematical expressions connected to the states represented in Fig.1.b,c,d are given as follows. The two double occupancies present in Fig.1.b provide the state

$$|b\rangle = (\hat{c}_{i,\uparrow}^\dagger \hat{c}_{i,\downarrow}^\dagger)(\hat{c}_{j,\uparrow}^\dagger \hat{c}_{j,\downarrow}^\dagger)|0\rangle, \quad (2)$$

where $|0\rangle$ represents the bare vacuum with no fermions present. The one double occupancy

and the two electrons with opposite spins depicted in Fig.1.c are described as

$$|c\rangle = (\hat{c}_{i,\uparrow}^\dagger \hat{c}_{i,\downarrow}^\dagger)[(\hat{c}_{j,\uparrow}^\dagger \hat{c}_{k,\downarrow}^\dagger) + (\hat{c}_{k,\uparrow}^\dagger \hat{c}_{j,\downarrow}^\dagger)]|0\rangle. \quad (3)$$

For the here presented situation we must have $j \neq k$ and $i \neq j, i \neq k$ respectively. Finally, in describing mathematically the four single occupancies presented in Fig.1.d, we have

$$|d\rangle = [(\hat{c}_{j,\uparrow}^\dagger \hat{c}_{i,\uparrow}^\dagger)(\hat{c}_{l,\downarrow}^\dagger \hat{c}_{k,\downarrow}^\dagger) + (\hat{c}_{j,\downarrow}^\dagger \hat{c}_{i,\downarrow}^\dagger)(\hat{c}_{l,\uparrow}^\dagger \hat{c}_{k,\uparrow}^\dagger)]|0\rangle, \quad (4)$$

where for the mathematical clarity, $j > i$ and $l > k$ is required (all sites being considered different).

2. The translation of the basic elements

In order to construct the base vectors, the basic wave vector elements are translated by the operation T . The T operator is considered a linear operator, hence the relation

$$T(A + B) = T(A) + T(B), \quad (5)$$

holds, where A and B represent particle configurations as depicted in Fig.1.b,c,d. Furthermore $T(A)$ means that the configuration of particles represented by A is translated to each site of the system (for $N = 16$, the translation is made 16 times), and the obtained contributions are all added (see Fig.2). The argument of the T operator contains usually four terms since the starting particle configuration is rotated by 180 degrees along the x, y , and z axes, and then the obtained contributions are added. Less than four terms in the T argument means that the mentioned rotations provide the same starting particle microconfiguration.

The translations are such effectuated that the relative inter-particle positions are all maintained, and arriving near the border of the system, the presence of the periodic boundary conditions are explicitly taken into account. The translations must be given at the level of the graphic plots, all the obtained 16 graphics must be added (see Fig.2), then each graphic must be transformed in a mathematical form according to the rules presented in Eqs.(2-4). At this point we must underline, that when the mathematical expression is written for a given graph, a negative sign may emerge in some cases given by the restrictions prescribed for Eqs.(2-4). For example, the last two terms in both rows of Fig.3 have mathematical expressions with negative sign. Similar reasons lead to sign changes obtained after rotations,

reason which generates the sign changes observed in some $T(\dots)$ arguments of base vectors presented in Figs.4-18, for example in the case of the second contributions of $|10\rangle$, or $|11\rangle$, etc. Furthermore, in some cases (see for example $|12\rangle$ or $|15\rangle$), the rotations provide the starting microconfiguration of particles, which leads to the decrease of the number of terms present in the argument of the T operation. The sign changes also lead in some cases to complete elimination of some possible particle microconfigurations.

3. The base wave vectors

Starting from the rules and operations described in details in Sec.II.B.1-2, the 85 base wave vectors can be explicitly constructed as given in Fig.4-18. These are denoted by $|n\rangle$, $n = 1, 2, \dots, 85$, and are all orthogonal vectors. As can be seen, the base wave vectors are obtained by subtracting two type of contributions. The first part of these is obtained from a starting particle microconfiguration (presented as first terms in the argument of the first operator T) which is rotated by 180 degrees along the x , y and z axes, and translated by the operation T after this step. The second (subtracted) part of the wave vectors is obtained from the first part by a rotation of the elements by 90 degree along the z axis.

C. The ground state wave function

First of all we are constructing a closed system of equations containing the base vectors described above.

This system of 85 equations is obtained as follows. i) We start from the most *condensed* and most *interacting* particle microconfiguration depicted by the base vector $|1\rangle$ (see Fig.4), which cannot be eliminated on physical grounds from the ground state. ii) By applying the Hamiltonian on the base vector $|1\rangle$ (see the first equation of Appendix A), we obtain as a result new base vectors ($|2\rangle$ and $|3\rangle$), which have the same symmetry properties as $|1\rangle$. iii) Since the Hamiltonian does not change the symmetry properties of the base vectors, one must further apply \hat{H} on the each resulting new base vector, up to the moment in which the obtained system of equations closes. This happens for the studied system at the 85th equation. The obtained closed system of equations is explicitly presented in Appendix A. The minimum energy solution of the presented system of equations represents the ground

state of the system.

III. PROPERTIES OF THE SOLUTIONS

A. The check of the ground state energies

First of all one can check that the system of equations presented in Appendix A indeed provides the ground state of the problem. The obtained minimum eigenvalues provided by Appendix A are

$$\begin{aligned}
u = 0.0, \quad E &= -12.0000000000000000, \\
u = 0.5, \quad E &= -11.9126285145094126, \\
u = 1.0, \quad E &= -11.8364690235642218, \\
u = 1.5, \quad E &= -11.7695877912829268, \\
u = 2.0, \quad E &= -11.7104580743242632, \\
u = 2.5, \quad E &= -11.6578605150913841, \\
u = 3.0, \quad E &= -11.6108110503797608, \\
u = 3.5, \quad E &= -11.5685079573602838, \\
u = 4.0, \quad E &= -11.5302924026297138,
\end{aligned} \tag{6}$$

which perfectly coincide with the numerically exact minimum eigenvalues obtained from exact numerical diagonalization in the 14 400 dimensional full Hilbert space (Sorella and Hubsh 2000). We note that the eigenvalues are given in t units.

B. The U independent eigenstates

The study of the solutions of the system of equations presented in Eq.(A1) leads to an extremely interesting observation, namely that almost half (exactly 40%) of the eigenstates are U independent, consequently are eigenstates of the non-interacting system as well. For example, the U independent 18 eigenvectors corresponding to zero eigenvalue are presented in Appendix B. Besides these, one has also 16 eigenfunctions corresponding to eigenvalues $\pm 4, \pm 8$, which are also U independent (see exemplification in Appendix B). The presence of such eigenstates is important for following reasons.

First of all, several authors observed (Galan and Verges 1991, Metzner and Vollhardt 1989) that weak coupling expansions often provide a good approximation for the Hubbard model at intermediate coupling. A possible explication for this is the emergence of a huge number of eigenstates in the spectrum of the Hubbard model with non-zero interaction, which are present in the non-interacting case as well in the spectrum of the model.

A second aspect which must be mentioned here is that the eigenstates related to zero eigenvalue presented in Appendix B are in fact many-body eigenstates of the kinetic energy term with no double occupancy. Such states apparently totally avoid double occupancy at no cost of energy, consequently are important in the study of the pairing mechanism in the Hubbard model (Cini, Perfetto and Stefanucci 2001, Cini and Stefanucci 2001, Perfetto and Cini 2004), and as seen from the presented results, easily emerge in the spectrum.

At this step we underline that contrary to the practice used in the theoretical studies up to this moment (Balzarotti et al. 2004), double occupancy avoiding eigenstates emerge not only at zero energy, but also at much lower energy values, closely situated to the ground state energy. We exemplify this statement by the eigenstate $|v_2\rangle$ presented in Eq.(B2), corresponding to energy -8 (in t units). These eigenstates being U independent, their energy remain unchanged if the strength of the on-site interaction is increased, while the ground-state energy increases if U is increased. Consequently, starting from these states, the contribution in the pairing mechanism of the kinetic energy eigenstates not containing double occupancy could be much more efficiently taken into account than considered up today. We note that this problem has interconnections to the problem of the kinetic energy driven superconductivity as well, strongly debated in the literature in the last period (Anderson 1995, Feng 2003, Hirsch 2004, Yokoyama et al. 2004). Indeed, experimentally is observed (Eckl, Hanke and Arrigoni 2003, van der Marel et al. 2003) that especially in *Bi* based cuprates, a decrease in the kinetic energy is observed at the superconducting transition, which contradicts the usual (for example BCS) pairing theories (Hirsch 2004). If the pairing process could be described on the states of the type $|v_2\rangle$, exemplified in our knowledge in exact terms for the first time in this paper, a such a decrease could be much better understood.

C. The U dependent eigenstates

The major part of the spectrum (60%) is U dependent. From the eigenstates entering in this category, there are states whose eigenvalue is exactly u , for example $|w\rangle = |8\rangle - |9\rangle + |39\rangle + |40\rangle - |41\rangle$. Such states are interesting for two reasons. First, a detection possibility of such states would lead to a direct experimental measurement of the U/t ratio. Second, since all the contributions in $|w\rangle$ have double occupancy $d = 1$, this vector is also an eigenstate of the interaction term with eigenvalue u , and in the same time, the eigenvector of the kinetic energy term with eigenvalue zero. Consequently, the study of the zero kinetic energy eigenvalue states must be given with care, since such type of states, as exemplified before, not necessarily are avoiding the double occupancy.

The remaining part of the deduced U dependent states, are not eigenstates of the kinetic energy, and the interacting part of \hat{H} separately, but are eigenvectors only for the sum of both, e.g. the whole Hamiltonian. As seen from Eq.(6), the corresponding eigenvalues have a smooth (less than linear) U dependence, and the ground states is obtained from this category.

D. The system generating the ground state

In order to enhance further developments and enlighten further creative thinking, we provide explicitly in Appendix A the equations generating the ground state, together with the explicit form of the base vectors providing the ground state (see Figs.4-18), hence describing the Hilbert space region in which this is placed. We mention that if the base vectors are defined only through the operation T , without subtracting two T components as shown in Figs.4-18, a system containing 176 equations arises. This however can be cast in two block diagonal components, one of which provides the equations presented in Appendix A.

IV. SUMMARY AND CONCLUSIONS

Driven by the aim to provide explicit expressions generating creative developments, we present exact analytical results describing the ground state of the two dimensional Hubbard model taken on an $L \times L = N$ square lattice at $N = 16$ (and periodic boundary conditions in both directions), containing $N_p = 4$ particles. The presented procedure allows the de-

scription of an arbitrary even L and N_p , and is based on symmetry adapted base vectors constructed in \mathbf{r} space. The Hamiltonian acting on the described base vectors provides a closed system of linear equations (whose number is 85 for $N = 16$ and $N_p = 4$), leading by its secular equation, through its minimum eigenvalue, to the ground state wave function and ground state energy of the system, deduced in a Hilbert subspace with almost three orders of magnitude smaller in dimensions than the full Hilbert space of the problem. The deduced explicit eigenstates characterize also other properties of the spectrum: i) The large number of eigenstates which remain eigenstates of the non-interacting Hamiltonian as well shows why weak coupling expansions often provide a good approximation for the Hubbard model at intermediate coupling. ii) Zero energy eigenstates of the kinetic energy term (\hat{H}_{kin}) which are eigenstates of the Hamiltonian as well show how energy increasing double occupancies can be avoided providing a possible support for the kinetic energy driven superconductivity. iii) Low energy eigenstates of the kinetic energy term which completely avoid double occupancy emphasize potentially new pairing possibilities in the low energy part of the spectrum in the context of the kinetic energy driven superconductivity. iv) Zero energy eigenstates of the kinetic energy term corresponding to double occupancy one underline that the zero energy \hat{H}_{kin} eigenstates must be handled with care, since as exemplified, these states not necessarily represent double occupancy avoiding states.

Similarly obtained solutions for higher L or N_p , corroborated with the study of the emerging changes in the system of equations describing the ground state, remain a challenge for future developments.

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APPENDIX A: THE LINEAR SYSTEM OF EQUATIONS CONTAINING THE GROUND STATE.

This Appendix presents the 85 linear equations describing the ground state of the studied system.

$$\hat{H}|1\rangle = 2u|1\rangle + |2\rangle - |3\rangle,$$

$$\hat{H}|2\rangle = |1\rangle + u|2\rangle + 8|4\rangle - |6\rangle - |7\rangle + |10\rangle + |11\rangle + 4|12\rangle,$$

$$\hat{H}|3\rangle = -8|1\rangle + u|3\rangle - 8|5\rangle + |6\rangle + |7\rangle - 2|8\rangle - 2|9\rangle - |10\rangle - |11\rangle - 2|13\rangle - 2|14\rangle - 4|15\rangle,$$

$$\hat{H}|4\rangle = |2\rangle + 2u|4\rangle - |17\rangle,$$

$$\hat{H}|5\rangle = -|3\rangle + |23\rangle,$$

$$\begin{aligned} \hat{H}|6\rangle = & -2|2\rangle + |3\rangle + u|6\rangle - |16\rangle + 2|17\rangle - 2|18\rangle - |19\rangle + |20\rangle + 2|21\rangle - |23\rangle + |24\rangle \\ & + |25\rangle + |26\rangle + |29\rangle + |31\rangle, \end{aligned}$$

$$\hat{H}|7\rangle = -2|2\rangle + |3\rangle + u|7\rangle - |16\rangle + 2|17\rangle + |20\rangle + 2|22\rangle - |23\rangle + |25\rangle + |26\rangle + |28\rangle + |30\rangle,$$

$$\hat{H}|8\rangle = -|3\rangle + u|8\rangle - |16\rangle - |21\rangle - |22\rangle + |23\rangle - |27\rangle - |33\rangle - |34\rangle,$$

$$\hat{H}|9\rangle = -|3\rangle + u|9\rangle - |16\rangle - |19\rangle - |20\rangle + |23\rangle - |24\rangle - |27\rangle - |32\rangle,$$

$$\begin{aligned} \hat{H}|10\rangle = & 2|2\rangle - |3\rangle + |16\rangle - 2|17\rangle - 2|18\rangle - |19\rangle + |23\rangle + |24\rangle - |25\rangle - |26\rangle - |28\rangle \\ & - |31\rangle - |32\rangle - 2|33\rangle, \end{aligned}$$

$$\hat{H}|11\rangle = 2|2\rangle - |3\rangle + |16\rangle - 2|17\rangle + |23\rangle - |25\rangle - |26\rangle - |29\rangle - |30\rangle - |32\rangle - 2|34\rangle,$$

$$\hat{H}|12\rangle = |2\rangle - |25\rangle,$$

$$\hat{H}|13\rangle = -|3\rangle - |16\rangle + |19\rangle + |23\rangle + |24\rangle - |27\rangle - |28\rangle - |29\rangle,$$

$$\hat{H}|14\rangle = -|3\rangle - |16\rangle + |23\rangle - |27\rangle - |30\rangle - |31\rangle,$$

$$\hat{H}|15\rangle = -|3\rangle - |26\rangle - |27\rangle,$$

$$\begin{aligned} \hat{H}|16\rangle = & -|6\rangle - |7\rangle - 2|8\rangle - 2|9\rangle + |10\rangle + |11\rangle - 2|13\rangle - 2|14\rangle + u|16\rangle - 8|36\rangle \\ & + 4|46\rangle - 4|54\rangle, \end{aligned}$$

$$\begin{aligned} \hat{H}|17\rangle = & -8|4\rangle + |6\rangle + |7\rangle - |10\rangle - |11\rangle + u|17\rangle + 4|36\rangle - 2|40\rangle - 2|41\rangle - 4|44\rangle \\ & - 4|47\rangle + 2|63\rangle + 2|64\rangle, \end{aligned}$$

$$\hat{H}|18\rangle = -|6\rangle - |10\rangle + u|18\rangle + |42\rangle - |56\rangle + |58\rangle,$$

$$\hat{H}|19\rangle = -|6\rangle - 2|9\rangle - |10\rangle + 2|13\rangle + u|19\rangle - 2|35\rangle - 4|39\rangle - 2|42\rangle - 2|55\rangle + |57\rangle + |58\rangle,$$

$$\begin{aligned}
\hat{H}|20\rangle &= |6\rangle + |7\rangle - 2|9\rangle + u|20\rangle - 2|35\rangle - 2|38\rangle - 4|40\rangle - 2|42\rangle - |43\rangle - |50\rangle - |51\rangle \\
&\quad + |57\rangle + 2|61\rangle + 2|65\rangle, \\
\hat{H}|21\rangle &= |6\rangle - |8\rangle + u|21\rangle - 2|41\rangle + |42\rangle - |43\rangle - |50\rangle - |56\rangle - |59\rangle, \\
\hat{H}|22\rangle &= |7\rangle - |8\rangle + u|22\rangle - 2|38\rangle - 2|41\rangle - |51\rangle - |59\rangle, \\
\hat{H}|23\rangle &= 8|5\rangle - |6\rangle - |7\rangle + 2|8\rangle + 2|9\rangle + |10\rangle + |11\rangle + 2|13\rangle + 2|14\rangle + 2|37\rangle \\
&\quad + 2|38\rangle + 2|43\rangle + 8|44\rangle - 4|45\rangle - 4|46\rangle, \\
\hat{H}|24\rangle &= |6\rangle - 2|9\rangle + |10\rangle + 2|13\rangle - 2|35\rangle - 4|39\rangle - 2|55\rangle - 2|56\rangle + |57\rangle - |58\rangle, \\
\hat{H}|25\rangle &= |6\rangle + |7\rangle - |10\rangle - |11\rangle - 8|12\rangle + 4|45\rangle + 4|48\rangle + 4|54\rangle - 2|59\rangle - 2|60\rangle \\
&\quad + 2|61\rangle + 2|62\rangle, \\
\hat{H}|26\rangle &= |6\rangle + |7\rangle - |10\rangle - |11\rangle - 4|15\rangle - 8|47\rangle + 4|48\rangle - |49\rangle - |50\rangle - |51\rangle - |52\rangle, \\
\hat{H}|27\rangle &= -2|8\rangle - 2|9\rangle - 2|13\rangle - 2|14\rangle - 4|15\rangle + 4|45\rangle - |49\rangle - |50\rangle - |51\rangle - |52\rangle \\
&\quad - 8|53\rangle - 4|54\rangle, \\
\hat{H}|28\rangle &= |7\rangle - |10\rangle - 2|13\rangle + 2|35\rangle - 2|38\rangle + 2|42\rangle - |43\rangle - |51\rangle - |52\rangle - |57\rangle \\
&\quad + 2|62\rangle + 4|63\rangle - 2|65\rangle, \\
\hat{H}|29\rangle &= |6\rangle - |11\rangle - 2|13\rangle + 2|35\rangle - 2|37\rangle - |43\rangle - |49\rangle - |50\rangle + 2|56\rangle - |57\rangle \\
&\quad + 2|62\rangle + 4|63\rangle - 2|65\rangle, \\
\hat{H}|30\rangle &= |7\rangle - |11\rangle - 2|14\rangle - 2|37\rangle - 2|38\rangle - |49\rangle - |51\rangle - 2|60\rangle + 4|64\rangle, \\
\hat{H}|31\rangle &= |6\rangle - |10\rangle - 2|14\rangle - 2|43\rangle - |50\rangle - |52\rangle - 2|60\rangle + 4|64\rangle, \\
\hat{H}|32\rangle &= -2|9\rangle - |10\rangle - |11\rangle - 2|35\rangle - 2|37\rangle - 4|40\rangle - |43\rangle - |49\rangle - |52\rangle - 2|56\rangle \\
&\quad + |57\rangle + 2|61\rangle + 2|65\rangle, \\
\hat{H}|33\rangle &= -|8\rangle - |10\rangle - 2|41\rangle - |42\rangle - |43\rangle - |52\rangle + |56\rangle - |59\rangle, \\
\hat{H}|34\rangle &= -|8\rangle - |11\rangle - 2|37\rangle - 2|41\rangle - |49\rangle - |59\rangle, \\
\hat{H}|35\rangle &= -|19\rangle - |20\rangle - |24\rangle + |28\rangle + |29\rangle - |32\rangle + |74\rangle - |75\rangle, \\
\hat{H}|36\rangle &= -|16\rangle + |17\rangle + 2u|36\rangle + |66\rangle, \\
\hat{H}|37\rangle &= |23\rangle - |29\rangle - |30\rangle - |32\rangle - 2|34\rangle + 2|67\rangle - |70\rangle + |71\rangle, \\
\hat{H}|38\rangle &= -|20\rangle - 2|22\rangle + |23\rangle - |28\rangle - |30\rangle + 2|67\rangle - |70\rangle + |71\rangle, \\
\hat{H}|39\rangle &= -|19\rangle - |24\rangle + u|39\rangle, \\
\hat{H}|40\rangle &= -|17\rangle - |20\rangle - |32\rangle + u|40\rangle - |66\rangle + |67\rangle - |69\rangle,
\end{aligned}$$

$$\begin{aligned}
\hat{H}|41\rangle &= -|17\rangle - |21\rangle - |22\rangle - |33\rangle - |34\rangle + u|41\rangle - |66\rangle + |67\rangle - |69\rangle, \\
\hat{H}|42\rangle &= |18\rangle - |19\rangle - |20\rangle + |21\rangle + |28\rangle - |33\rangle + u|42\rangle + |75\rangle - |76\rangle, \\
\hat{H}|43\rangle &= -|20\rangle - 2|21\rangle + 2|23\rangle - |28\rangle - |29\rangle - 2|31\rangle - |32\rangle - 2|33\rangle + 4|67\rangle - 2|70\rangle + 2|71\rangle, \\
\hat{H}|44\rangle &= -|17\rangle + |23\rangle + |67\rangle, \\
\hat{H}|45\rangle &= -|23\rangle + |25\rangle + |27\rangle + |70\rangle, \\
\hat{H}|46\rangle &= |16\rangle - |23\rangle - |71\rangle, \\
\hat{H}|47\rangle &= -|17\rangle - |26\rangle - |69\rangle, \\
\hat{H}|48\rangle &= |25\rangle + |26\rangle - |72\rangle, \\
\hat{H}|49\rangle &= -|26\rangle - |27\rangle - |29\rangle - |30\rangle - |32\rangle - 2|34\rangle - 2|68\rangle - 2|69\rangle - |70\rangle + |72\rangle - |73\rangle, \\
\hat{H}|50\rangle &= -|20\rangle - 2|21\rangle - |26\rangle - |27\rangle - |29\rangle - |31\rangle - 2|68\rangle - 2|69\rangle - |70\rangle + |72\rangle \\
&\quad - |73\rangle + |74\rangle - |75\rangle - 2|76\rangle, \\
\hat{H}|51\rangle &= -|20\rangle - 2|22\rangle - |26\rangle - |27\rangle - |28\rangle - |30\rangle - 2|68\rangle - 2|69\rangle - |70\rangle + |72\rangle - |73\rangle, \\
\hat{H}|52\rangle &= -|26\rangle - |27\rangle - |28\rangle - |31\rangle - |32\rangle - 2|33\rangle - 2|68\rangle - 2|69\rangle - |70\rangle + |72\rangle \\
&\quad - |73\rangle - |74\rangle + |75\rangle + 2|76\rangle, \\
\hat{H}|53\rangle &= -|27\rangle - |68\rangle, \\
\hat{H}|54\rangle &= -|16\rangle + |25\rangle - |27\rangle - |73\rangle, \\
\hat{H}|55\rangle &= -|19\rangle - |24\rangle + |74\rangle + |75\rangle, \\
\hat{H}|56\rangle &= -|18\rangle - |21\rangle - |24\rangle + |29\rangle - |32\rangle + |33\rangle + |74\rangle + |76\rangle, \\
\hat{H}|57\rangle &= |19\rangle + |20\rangle + |24\rangle - |28\rangle - |29\rangle + |32\rangle - |74\rangle - |75\rangle, \\
\hat{H}|58\rangle &= 2|18\rangle + |19\rangle - |24\rangle - |74\rangle + |75\rangle + 2|76\rangle, \\
\hat{H}|59\rangle &= -|21\rangle - |22\rangle - |25\rangle - |33\rangle - |34\rangle - |70\rangle + |72\rangle + |73\rangle, \\
\hat{H}|60\rangle &= -|25\rangle - |30\rangle - |31\rangle - |70\rangle + |72\rangle + |73\rangle, \\
\hat{H}|61\rangle &= |20\rangle + |25\rangle + |32\rangle + |70\rangle - |72\rangle - |73\rangle - |74\rangle - |75\rangle, \\
\hat{H}|62\rangle &= |25\rangle + |28\rangle + |29\rangle + |70\rangle - |72\rangle - |73\rangle + |74\rangle + |75\rangle, \\
\hat{H}|63\rangle &= |17\rangle + |28\rangle + |29\rangle + |66\rangle - |67\rangle + |69\rangle, \\
\hat{H}|64\rangle &= |17\rangle + |30\rangle + |31\rangle + |66\rangle - |67\rangle + |69\rangle, \\
\hat{H}|65\rangle &= |20\rangle - |28\rangle - |29\rangle + |32\rangle, \\
\hat{H}|66\rangle &= 4|36\rangle - 2|40\rangle - 2|41\rangle + 2|63\rangle + 2|64\rangle + u|66\rangle + 4|78\rangle - 4|85\rangle,
\end{aligned}$$

$$\begin{aligned}
\hat{H}|67\rangle &= 2|37\rangle + 2|38\rangle + 2|40\rangle + 2|41\rangle + 2|43\rangle + 4|44\rangle - 2|63\rangle - 2|64\rangle + 16|77\rangle \\
&\quad - 4|78\rangle + 4|79\rangle, \\
\hat{H}|68\rangle &= -|49\rangle - |50\rangle - |51\rangle - |52\rangle - 4|53\rangle - 8|80\rangle + 4|81\rangle, \\
\hat{H}|69\rangle &= -2|40\rangle - 2|41\rangle - 4|47\rangle - |49\rangle - |50\rangle - |51\rangle - |52\rangle + 2|63\rangle + 2|64\rangle - 4|79\rangle \\
&\quad - 8|80\rangle - 4|85\rangle, \\
\hat{H}|70\rangle &= -2|37\rangle - 2|38\rangle - 2|43\rangle + 4|45\rangle - |49\rangle - |50\rangle - |51\rangle - |52\rangle - 2|59\rangle - 2|60\rangle \\
&\quad + 2|61\rangle + 2|62\rangle - 8|79\rangle + 8|82\rangle - 4|84\rangle, \\
\hat{H}|71\rangle &= 2|37\rangle + 2|38\rangle + 2|43\rangle - 4|46\rangle - 8|78\rangle + 4|84\rangle, \\
\hat{H}|72\rangle &= -4|48\rangle + |49\rangle + |50\rangle + |51\rangle + |52\rangle + 2|59\rangle + 2|60\rangle - 2|61\rangle - 2|62\rangle \\
&\quad - 8|81\rangle - 8|82\rangle, \\
\hat{H}|73\rangle &= -|49\rangle - |50\rangle - |51\rangle - |52\rangle - 4|54\rangle + 2|59\rangle + 2|60\rangle - 2|61\rangle - 2|62\rangle \\
&\quad + 4|84\rangle - 8|85\rangle, \\
\hat{H}|74\rangle &= 2|35\rangle + |50\rangle - |52\rangle + 2|55\rangle + 2|56\rangle - |57\rangle - |58\rangle - 2|61\rangle + 2|62\rangle + 4|83\rangle, \\
\hat{H}|75\rangle &= 2|35\rangle + 2|42\rangle - |50\rangle + |52\rangle + 2|55\rangle - |57\rangle + |58\rangle - 2|61\rangle + 2|62\rangle + 4|83\rangle, \\
\hat{H}|76\rangle &= -|42\rangle - |50\rangle + |52\rangle + |56\rangle + |58\rangle, \\
\hat{H}|77\rangle &= |67\rangle, \\
\hat{H}|78\rangle &= |66\rangle - |67\rangle - |71\rangle, \\
\hat{H}|79\rangle &= |67\rangle - |69\rangle - |70\rangle, \\
\hat{H}|80\rangle &= -|68\rangle - |69\rangle, \\
\hat{H}|81\rangle &= |68\rangle - |72\rangle, \\
\hat{H}|82\rangle &= |70\rangle - |72\rangle, \\
\hat{H}|83\rangle &= |74\rangle + |75\rangle, \\
\hat{H}|84\rangle &= -|70\rangle + |71\rangle + |73\rangle, \\
\hat{H}|85\rangle &= -|66\rangle - |69\rangle - |73\rangle.
\end{aligned} \tag{A1}$$

APPENDIX B: THE U INDEPENDENT EIGENVECTORS CORRESPONDING TO ZERO EIGENVALUE.

The U independent eigenvalues corresponding to zero eigenvalues are the following

$$\begin{aligned}
|\theta_1\rangle &= |35\rangle + |57\rangle, \\
|\theta_2\rangle &= |55\rangle + |57\rangle - |65\rangle, \\
|\theta_3\rangle &= |37\rangle - |38\rangle + |51\rangle - |49\rangle, \\
|\theta_4\rangle &= |60\rangle + |61\rangle - |63\rangle + |64\rangle - |65\rangle + |83\rangle, \\
|\theta_5\rangle &= |60\rangle + |62\rangle - |63\rangle + |64\rangle - |83\rangle, \\
|\theta_6\rangle &= |77\rangle + |78\rangle - |80\rangle - |81\rangle + |82\rangle + |84\rangle + |85\rangle, \\
|\theta_7\rangle &= -|77\rangle + |79\rangle - |80\rangle - |81\rangle + |82\rangle, \\
|\theta_8\rangle &= |10\rangle - |11\rangle + |49\rangle - |52\rangle + |58\rangle, \\
|\theta_9\rangle &= |13\rangle - |14\rangle + |63\rangle - |64\rangle + |55\rangle - |83\rangle, \\
|\theta_{10}\rangle &= -|5\rangle + |15\rangle + |44\rangle - |47\rangle - |53\rangle - |77\rangle + |80\rangle, \\
|\theta_{11}\rangle &= |14\rangle - |15\rangle + |46\rangle - |48\rangle - |60\rangle + |84\rangle, \\
|\theta_{12}\rangle &= |44\rangle + |45\rangle - |47\rangle - |48\rangle + |53\rangle - 2|77\rangle + |79\rangle + |81\rangle, \\
|\theta_{13}\rangle &= |29\rangle - |28\rangle + |30\rangle - |31\rangle + |75\rangle - |74\rangle + 2|33\rangle - 2|34\rangle - 2|76\rangle, \\
|\theta_{14}\rangle &= -|45\rangle + |46\rangle + |54\rangle + 2|77\rangle - 2|53\rangle - 2|79\rangle + 2|80\rangle + |84\rangle, \\
|\theta_{15}\rangle &= |37\rangle - |44\rangle + |47\rangle - |49\rangle - |77\rangle + |53\rangle + |80\rangle - |82\rangle - |84\rangle, \\
|\theta_{16}\rangle &= |43\rangle - 2|37\rangle + 2|49\rangle - |50\rangle - |52\rangle, \\
|\theta_{17}\rangle &= |10\rangle - 2|12\rangle - |15\rangle - 2|44\rangle - |45\rangle - |48\rangle - |52\rangle + |54\rangle + |58\rangle + 2|77\rangle + 2|80\rangle, \\
|\theta_{18}\rangle &= -2|5\rangle - 2|15\rangle + 2|10\rangle + 2|11\rangle - 8|12\rangle - |43\rangle - 8|47\rangle - 8|48\rangle - 2|49\rangle + |50\rangle \\
&\quad - |52\rangle + 4|54\rangle + 2|58\rangle + 4|77\rangle + 12|80\rangle + 8|81\rangle - 2|82\rangle + 2|84\rangle, \tag{B1}
\end{aligned}$$

For exemplification we present also two U independent eigenfunctions corresponding to non-zero eigenvalues, namely $+8$ (the vector $|v_1\rangle$), and -8 (the vector $|v_2\rangle$).

$$\begin{aligned}
|v_1\rangle &= |5\rangle - |15\rangle + |23\rangle + |26\rangle + |27\rangle + |37\rangle + |38\rangle + |43\rangle + 2|44\rangle - |46\rangle - 2|47\rangle + |48\rangle \\
&\quad - |49\rangle - |50\rangle - |51\rangle - |52\rangle - 2|53\rangle - |54\rangle + 2|67\rangle + 2|68\rangle + 2|69\rangle + |71\rangle - |72\rangle \\
&\quad + |73\rangle + 4|77\rangle - 2|78\rangle - 4|80\rangle + 2|81\rangle + |82\rangle + |84\rangle - 2|85\rangle,
\end{aligned}$$

$$\begin{aligned}
|v_2\rangle = & |5\rangle - |15\rangle - |23\rangle - |26\rangle - |27\rangle + |37\rangle + |38\rangle + |43\rangle + 2|44\rangle - |46\rangle - 2|47\rangle + |48\rangle \\
& - |49\rangle - |50\rangle - |51\rangle - |52\rangle - 2|53\rangle - |54\rangle - 2|67\rangle - 2|68\rangle - 2|69\rangle - |71\rangle + |72\rangle \\
& - |73\rangle + 4|77\rangle - 2|78\rangle - 4|80\rangle + 2|81\rangle + |82\rangle + |84\rangle - 2|85\rangle,
\end{aligned} \tag{B2}$$

-
- ACQUARONE, M., CUCCO, M., NOCE, C., and ROMANO, A., 1999, *Physica* **B261**, 725.
- ANDERSON, P., W., 1995, *Science* **268**, 1154.
- BAERISWYL, D., CAMPBELL, D., K., CARMELO, J., M., P., GUINEA, F., and LOUIS, E., Eds., 1995, *The Hubbard Model, Its Physics and Mathematical Physics*, NATO ASI Series B, vol. 343, Plenum Press, New York and London.
- BALZAROTTI, A., et. al., 2004, cond-mat/0411101.
- BOYACI, H., GEDIK, Z., and KULIK, I., O., 2000, *Jour. of Supercond.* **13**, 1031; *ibid.* 2001, **14**, 133.
- BRUUS, H., and D'AURIAC, J., C., A., 1996, *Europhys. Lett.* **35**, 321.
- BRUUS, H., and D'AURIAC, J., C., A., 1997, *Phys. Rev.* **B55**, 9142.
- BUSSER, C., A., MORENO, A., and DAGOTTO, E., 2004, *Phys. Rev.* **B70**, 035402.
- CHEN, L., and MEI, C., 1989, *Phys. Rev.* **B39**, 9006.
- CHIAPPE, G., BUSSE, C., ANDA, E., V., and FERRARI, V., 1999, *Jour. of Phys.* **C11**, 5237.
- CINI, M., PERFETTO, E., and STEFANUCCI, G., 2001, *Eur. Phys. Jour.* **B20**, 91.
- CINI, M., and STEFANUCCI, G., 2001, *Jour. of Phys.* **C13**, 1279.
- ECKL, T., HANKE, W., and ARRIGONI, E., 2003, *Phys. Rev.* **B68**, 014505.
- EROLE, J., BATISTA, C., D., and ALIGIA, A., A., 1999, *Phys. Rev.* **B59**, 14092.
- FABRIZIO, M., PAROLA, A., and TOSATTI, E., 1991, *Phys. Rev.* **B44**, 1033.
- FALICOV, L., M., and PROETTO, C., R., 1993, *Phys. Rev.* **B47**, 14407.
- FANO, G., ORTOLANI, F., and PAROLA, A., 1992, *Phys. Rev.* **B46**, 1048.
- FENG, S., 2003, *Phys. Rev.* **B68**, 184501.
- GALAN, J., and VERGES, J., A., 1991, *Phys. Rev.* **B44**, 10093.
- HALFPAP, O., 2001, *Annalen der Physik* **10**, 623.
- HELBERG, C., S., 2001, *Jour. Appl. Phys.* **89**, 6627.
- HIRSCH, J., E., 2000, *Physica* **C341-348**, 213; and *ibid* 1992, **C199**, 305.

- HUBBARD, J., 1963, Proc. R. Soc. London **A276**, 238.
- KOCHERESHKO, V., P., et al., 2003, Physica **E17**, 197.
- LIEB, E., H., 1994, in *Proceedings of the XIth International Congress of Mathematical Physics*, Paris, edited by D. Iagolnitzer, International Press, Paris, pp. 392.
- LIEB, E., H., and WU, F., Y., 1968, Phys. Rev. Lett. **20**, 1445.
- LOUIS, E., GALAN, J., GUINEA, F., VERGES, J., A., and FERRER, J., 1992, Phys. Stat. Solidi **B173**, 715.
- MAKSYM., P., A., IMAMURA, H., MALLON, G., P., and AOKI, H., 2000, Jour. of Phys. **C12**, R299.
- MAREL, VAN, DER, D., MOLEGRAAF, H., J., A., PRESURA, C., and SANTOSO, I., 2003, cond-mat/0302169.
- MATTIS, D., C., 1993, *An Encyclopedia of Exactly Solved Models in One Dimension: The Many-Body Problem*, World Scientific, London.
- MEI, C., and CHEN, L., 1988, Zeit.Phys. **B72**, 429.
- METZNER, W., and VOLLHARDT, D., 1989, Phys. Rev. **B39**, 4462.
- OKADA, K., 2004, Jour. Phys. Soc. Jpn. **73**, 1681.
- PAPAVASSILIOU, G., C., and YARTSEV, V., M., 1992, Chem. Phys. Lett., **200**, 209.
- PAROLA, A., SORELLA, S., PARRINELLO, M., and TOSATTI, E., 1990, in *Dynamics of Magnetic Fluctuations in High Temperature Superconductors*, edited by G. Reiner, P. Horsch, and G. Psaltakis, Plenum, New York.
- PAROLA, A., SORELLA, S., PARRINELLO, M., and TOSATTI, E., 1991, Phys. Rev. **B43**, 6190.
- PERFETTO, E., and CINI, M., 2004, Jour. of Phys. **C16**, 4845.
- SACKETT, C., A., 2000, Nature **404**, 256.
- SAITO, G., HIRATE, S., NISHIMURA, K., and YAMOCHI, H., 2001, Jour. of Mater. Chem. **11**, 723.
- SORELLA, S., and HUBSH, A., 2000, *We would like to thank to S. Sorella and A. Hubsh for providing us numerical exact diagonalization results for the test of the deduced system of equations.*
- SRINITIWARAWONG, C., and GEHRING, G., A., 2002, Jour. of Phys. **C14**, 11589.
- TJEMBERG, O., 1998, Jour. Math. Phys. **39**, 6416.
- YOKOYAMA, H., TANAKA, Y., OGATA, M., and TSUCHIURA, H., 2004, Jour. Phys. Soc. Jpn. **73**, 1119.

- ZHANG, N., G., and HENLEY, C., L., 2004, Eur. Phys. Jour. **B38**, 409.
- ZHU, H., Y., and JIANG, Y., S., 1993, Acta Chim. Sinica **51**, 527.

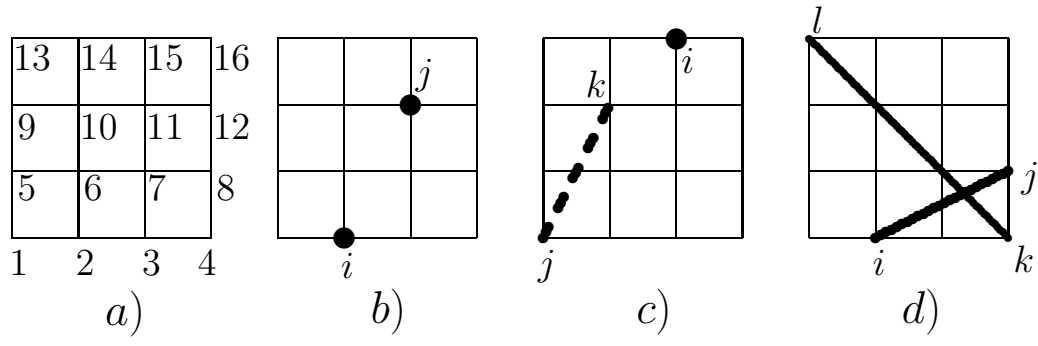


FIG. 1: The site numbering for a 4×4 cluster, and the possible particle displacements for four electrons.

$$\begin{aligned}
T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}\right) &= \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \\
&+ \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \dots
\end{aligned}$$

FIG. 2: The effect of the translation operator T . The result in the right hand side contains 16 terms, from which, the first 8 are plotted in the figure.

$$\begin{aligned}
T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \leftarrow & & \\ \hline & & \\ \hline \end{array}\right) &= \begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \leftarrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \leftarrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \leftarrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \leftarrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \leftarrow & & \\ \hline & & \\ \hline \end{array} \\
&+ \begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \leftarrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \leftarrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \leftarrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \leftarrow & & \\ \hline & & \\ \hline \end{array} + \dots
\end{aligned}$$

FIG. 3: The effect of the translation operator T leading to sign changes in the mathematical expressions.

$$|1\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \bullet & \bullet & \\ \hline \end{array} \right) - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \end{array} \right),$$

$$|2\rangle = T\left(\begin{array}{|c|c|c|} \hline \bullet & & \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \bullet & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \right),$$

$$|3\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \bullet & \vdots & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline \vdots & & \\ \hline \vdots & \bullet & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & \vdots & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline \vdots & \bullet & \\ \hline \end{array} \right)$$

$$-T\left(\begin{array}{|c|c|c|} \hline \vdots & \vdots & \\ \hline \vdots & \vdots & \\ \hline \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \vdots & \vdots & \\ \hline \vdots & \vdots & \\ \hline \vdots & \bullet & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \bullet & \vdots & \\ \hline \vdots & \vdots & \\ \hline \vdots & \vdots & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \bullet & \vdots & \\ \hline \vdots & \vdots & \\ \hline \vdots & \vdots & \\ \hline \end{array} \right)$$

$$|4\rangle = \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \bullet & & \bullet \\ \hline\end{array}\right) - \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline \bullet & & \\ \hline & & \\ \hline & & \\ \hline \bullet & & \\ \hline\end{array}\right), \quad |5\rangle = T\left(\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \times & & \\ \hline\end{array}\right),$$

$$|6\rangle = T \left(\begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \right) - T \left(\begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \right),$$

FIG. 4: Base wave vectors $|1\rangle - |6\rangle$.

FIG. 5: Base wave vectors $|7\rangle - |11\rangle$.

$$|12\rangle = \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}\right) - \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline \blacksquare & \square & \square \\ \hline \square & \square & \square \\ \hline \blacksquare & \square & \square \\ \hline \end{array}\right),$$

[illegible]

$$|14\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \diagup & \diagup & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \diagdown & \diagdown & \\ \hline \end{array} \right) - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline \diagup & \diagup & \\ \hline \diagup & \diagup & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & & \\ \hline \diagdown & \diagdown & \\ \hline \diagdown & \diagdown & \\ \hline \end{array} \right),$$

$$|15\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \textcolor{black}{\rule{0.5pt}{1cm}} & \textcolor{black}{\rule{0.5pt}{1cm}} & \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \textcolor{black}{\rule{0.5pt}{1cm}} & & \\ \hline \end{array}\right),$$

$$|16\rangle = T\left(\begin{array}{c} \bullet \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \bullet \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \bullet \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \bullet \end{array} \right) - T\left(\begin{array}{c} \vdots \\ \vdots \\ \bullet \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \bullet \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \bullet \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \bullet \end{array} \right),$$

$$|17\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \bullet \\ \hline & & \\ \hline \bullet & & \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|} \hline \bullet & & \\ \hline & & \\ \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline & & \\ \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}\right),$$

$$|18\rangle = T(\text{diagram 1} + \text{diagram 2}) - T(\text{diagram 3} + \text{diagram 4}),$$

FIG. 6: Base wave vectors $|12\rangle - |18\rangle$.

FIG. 7: Base wave vectors $|19\rangle - |23\rangle$.

FIG. 8: Base wave vectors $|24\rangle - |27\rangle$.

$$\begin{aligned}
|32\rangle &= T \left(\begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \\ + \\ \text{Diagram 3} \\ + \\ \text{Diagram 4} \end{array} \right) - T \left(\begin{array}{c} \text{Diagram 5} \\ - \\ \text{Diagram 6} \\ - \\ \text{Diagram 7} \\ - \\ \text{Diagram 8} \end{array} \right), \\
|33\rangle &= T \left(\begin{array}{c} \text{Diagram 9} \\ + \\ \text{Diagram 10} \end{array} \right) - T \left(\begin{array}{c} \text{Diagram 11} \\ + \\ \text{Diagram 12} \end{array} \right), \\
|34\rangle &= T \left(\begin{array}{c} \text{Diagram 13} \\ + \\ \text{Diagram 14} \end{array} \right) - T \left(\begin{array}{c} \text{Diagram 15} \\ + \\ \text{Diagram 16} \end{array} \right), \\
|35\rangle &= T \left(\begin{array}{c} \text{Diagram 17} \\ + \\ \text{Diagram 18} \\ + \\ \text{Diagram 19} \\ + \\ \text{Diagram 20} \end{array} \right), \\
|36\rangle &= T \left(\begin{array}{c} \text{Diagram 21} \end{array} \right) - T \left(\begin{array}{c} \text{Diagram 22} \end{array} \right), \\
|37\rangle &= T \left(\begin{array}{c} \text{Diagram 23} \\ + \\ \text{Diagram 24} \\ + \\ \text{Diagram 25} \\ + \\ \text{Diagram 26} \end{array} \right), \\
|38\rangle &= T \left(\begin{array}{c} \text{Diagram 27} \\ + \\ \text{Diagram 28} \\ + \\ \text{Diagram 29} \\ + \\ \text{Diagram 30} \end{array} \right),
\end{aligned}$$

FIG. 10: Base wave vectors $|32\rangle - |38\rangle$.

$$|39\rangle = T\left(\begin{array}{ccc} \bullet & & \\ | & & \\ | & & \\ | & & \\ | & & \\ \bullet & & \end{array}\right) - T\left(\begin{array}{ccc} & & \\ & & \\ & & \\ & & \\ & \bullet & \\ \bullet & \bullet & \bullet \end{array}\right),$$

$$|40\rangle = T\left(\begin{array}{|c|c|c|} \hline \bullet & & \\ \hline \bullet & & \\ \hline \bullet & \bullet & \\ \hline \bullet & & \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|} \hline & \bullet & \\ \hline & & \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \end{array}\right),$$

$$|41\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \bullet \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|} \hline & & \bullet \\ \hline & & \\ \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}\right),$$

$$|42\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \bullet \\ \hline & & \bullet \\ \hline \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \bullet \\ \hline & & \bullet \\ \hline \bullet & & \\ \hline \end{array} \right) - T\left(\begin{array}{|c|c|c|} \hline & & \bullet \\ \hline & & \bullet \\ \hline \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \bullet \\ \hline & \bullet & \\ \hline \bullet & & \\ \hline \end{array} \right),$$

$$|43\rangle = T\left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) - T\left(\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right),$$

$$|44\rangle = \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline \bullet & & \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \end{array}\right) - \left(-\frac{1}{2}\right)T\left(\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline \bullet & \bullet & \\ \hline \bullet & \bullet & \\ \hline \end{array}\right),$$

$$|45\rangle = \frac{1}{2}T\left(\begin{array}{|c|c|c|c|}\hline \diagdown & & & \\ \hline & & & \\ \hline \diagdown & & & \\ \hline & & & \\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline & \diagup & & \\ \hline & & & \\ \hline & \diagup & & \\ \hline & & & \\ \hline\end{array}\right) - \frac{1}{2}T\left(\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline \diagdown & & \diagdown & \\ \hline & & & \\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline \diagup & & \diagup & \\ \hline & & & \\ \hline\end{array}\right),$$

FIG. 11: Base wave vectors $|39\rangle - |45\rangle$.

$$|46\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}\right),$$

$$|47\rangle = \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline \bullet & \bullet & \\ \hline \bullet & \bullet & \\ \hline \bullet & \bullet & \\ \hline \end{array}\right) - \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline & & \\ \hline \bullet & \bullet & \\ \hline \bullet & \bullet & \\ \hline \end{array}\right),$$

$$|48\rangle = T\left(\begin{array}{|c|c|c|} \hline \text{thick} & & \\ \hline & & \\ \hline & \text{thick} & \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \text{thick} & \text{thick} & \\ \hline \end{array}\right),$$

$$|49\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & \diagdown & \\ \hline | & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & \diagup & \\ \hline & & | \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \diagup \\ \hline | & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline \diagdown & & \\ \hline & & | \\ \hline \end{array} \right) \\ - T\left(\begin{array}{|c|c|c|} \hline | & & \\ \hline & & \diagup \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \diagup \\ \hline & & | \\ \hline & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & & \diagdown \\ \hline & & \\ \hline | & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & & \\ \hline \diagdown & & | \\ \hline & & \\ \hline \end{array} \right),$$

$$|50\rangle = T\left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) - T\left(\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right),$$

FIG. 12: Base wave vectors $|46\rangle - |50\rangle$.

$$\begin{aligned}
|51\rangle &= T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & \nearrow & \\ \hline \downarrow & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline \searrow & & \downarrow \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \nearrow & \downarrow \\ \hline & & \\ \hline \downarrow & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \searrow & \\ \hline \downarrow & & \\ \hline & & \\ \hline \end{array} \right) \\
&\quad - T\left(\begin{array}{|c|c|c|} \hline \nearrow & & \\ \hline \downarrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \rightarrow & \\ \hline \nearrow & & \\ \hline & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline \rightarrow & & \\ \hline \searrow & & \\ \hline & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & \searrow & \\ \hline & \downarrow & \\ \hline & & \\ \hline \end{array} \right), \\
|52\rangle &= T\left(\begin{array}{|c|c|c|} \hline \downarrow & & \\ \hline & \searrow & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \nearrow & \downarrow \\ \hline \nearrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \downarrow & \nearrow & \\ \hline & & \\ \hline \downarrow & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \searrow & & \downarrow \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) \\
&\quad - T\left(\begin{array}{|c|c|c|} \hline & \nearrow & \\ \hline \downarrow & & \\ \hline \downarrow & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \rightarrow & \rightarrow & \\ \hline \nearrow & & \\ \hline & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline \searrow & & \\ \hline \downarrow & & \\ \hline \downarrow & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline \rightarrow & \rightarrow & \\ \hline & \searrow & \\ \hline & & \\ \hline \end{array} \right), \\
|53\rangle &= \frac{1}{2}T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \downarrow & & \downarrow \\ \hline \end{array} \right) - \frac{1}{2}T\left(\begin{array}{|c|c|c|} \hline \rightarrow & & \\ \hline & & \\ \hline \downarrow & & \\ \hline \end{array} \right), \\
|54\rangle &= T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline \nearrow & & \searrow \\ \hline & & \\ \hline \end{array} \right) - T\left(- \begin{array}{|c|c|c|} \hline \searrow & & \\ \hline \nearrow & & \\ \hline & & \\ \hline \end{array} \right), \\
|55\rangle &= T\left(\begin{array}{|c|c|c|} \hline & \downarrow & \\ \hline \downarrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \downarrow & & \\ \hline & \downarrow & \\ \hline & & \\ \hline \end{array} \right) - T\left(\begin{array}{|c|c|c|} \hline & \rightarrow & \\ \hline \downarrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \rightarrow & & \\ \hline \downarrow & & \\ \hline & & \\ \hline \end{array} \right), \\
|56\rangle &= T\left(\begin{array}{|c|c|c|} \hline & & \downarrow \\ \hline \searrow & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \nearrow & \\ \hline \nearrow & & \downarrow \\ \hline & & \\ \hline \end{array} \right) - T\left(\begin{array}{|c|c|c|} \hline \nearrow & & \\ \hline \downarrow & & \\ \hline \downarrow & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \rightarrow & \nearrow & \\ \hline & & \\ \hline \downarrow & & \\ \hline \end{array} \right),
\end{aligned}$$

FIG. 13: Base wave vectors $|51\rangle - |56\rangle$.

FIG. 14: Base wave vectors $|57\rangle - |62\rangle$.

$$|63\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline \diagup & & \\ \hline \diagup & & \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline \diagup & \diagup & \\ \hline \diagup & \diagup & \\ \hline \end{array}\right),$$

$$|64\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & \diagup & \\ \hline \diagup & \diagup & \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline \diagup & \diagup & \\ \hline \diagup & \diagup & \\ \hline \end{array}\right),$$

$$|65\rangle = T\left(\begin{array}{|c|c|c|} \hline & \diagup & \\ \hline \diagdown & & \\ \hline \diagdown & & \\ \hline \end{array}\right) + \begin{array}{|c|c|c|} \hline \diagup & & \\ \hline \diagup & & \\ \hline \diagup & & \\ \hline \end{array} - T\left(\begin{array}{|c|c|c|} \hline & \diagup & \\ \hline \diagup & \diagup & \\ \hline \diagup & \diagup & \\ \hline \end{array}\right) + \begin{array}{|c|c|c|} \hline & \diagdown & \\ \hline \diagdown & \diagdown & \\ \hline \diagdown & \diagdown & \\ \hline \end{array},$$

$$|66\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \bullet \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \end{array}\right) + \begin{array}{|c|c|c|} \hline \bullet & & \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \end{array} - T\left(\begin{array}{|c|c|c|} \hline & & \bullet \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \end{array}\right) + \begin{array}{|c|c|c|} \hline \bullet & & \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \end{array},$$

$$|67\rangle = T\left(-\begin{array}{|c|c|c|} \hline & & \\ \hline \diagup & \diagup & \\ \hline \diagup & \diagup & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & & \\ \hline \diagdown & \diagdown & \\ \hline \diagdown & \diagdown & \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline \diagup & \diagup & \\ \hline \diagup & \diagup & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline \diagdown & \diagdown & \\ \hline \diagdown & \diagdown & \\ \hline \end{array}\right),$$

$$|68\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}\right) + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}\right) + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array},$$

$$|69\rangle = T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline \diagup & & \\ \hline \diagup & & \\ \hline \end{array}\right) + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} - T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline \diagup & & \\ \hline \diagup & & \\ \hline \end{array}\right) + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array},$$

FIG. 15: Base wave vectors $|63\rangle - |69\rangle$.

$$\begin{aligned}
|74\rangle &= T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline \text{diag} & & \\ \hline & \text{hline} & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \text{hline} & & \\ \hline & \text{diag} & \\ \hline & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & \text{hline} & \\ \hline \text{diag} & & \\ \hline & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & \text{diag} & \\ \hline & & \text{hline} \\ \hline & & \\ \hline \end{array} \right) \\
&\quad - T\left(\begin{array}{|c|c|c|} \hline & & \text{vline} \\ \hline \text{diag} & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline \text{vline} & & \text{diag} \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \text{diag} & \\ \hline \text{vline} & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \text{diag} & \\ \hline & & \text{vline} \\ \hline & & \\ \hline \end{array} \right), \\
|75\rangle &= T\left(\begin{array}{|c|c|c|} \hline & & \\ \hline \text{diag} & & \\ \hline \text{hline} & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \text{hline} & \\ \hline \text{diag} & & \\ \hline & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline \text{hline} & & \\ \hline & \text{diag} & \\ \hline & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & \text{diag} & \\ \hline & & \text{hline} \\ \hline & & \\ \hline \end{array} \right) \\
&\quad - T\left(\begin{array}{|c|c|c|} \hline & \text{diag} & \\ \hline \text{vline} & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \text{vline} \\ \hline \text{diag} & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline \text{vline} & & \text{diag} \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \text{diag} & \\ \hline & & \text{vline} \\ \hline & & \\ \hline \end{array} \right), \\
|76\rangle &= T\left(\begin{array}{|c|c|c|} \hline \text{vline} & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \text{hline} & & \text{vline} \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) - T\left(\begin{array}{|c|c|c|} \hline \text{vline} & & \\ \hline & & \\ \hline \text{hline} & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & \text{hline} & \text{vline} \\ \hline & & \\ \hline & & \\ \hline \end{array} \right), \\
|77\rangle &= \frac{1}{4}T\left(\begin{array}{|c|c|c|} \hline \text{diag} & \text{diag} & \\ \hline \text{diag} & \text{diag} & \\ \hline \text{diag} & \text{diag} & \\ \hline \end{array} \right), \\
|78\rangle &= \frac{1}{2}T\left(\begin{array}{|c|c|c|} \hline & \text{diag} & \\ \hline \text{diag} & \text{diag} & \\ \hline \text{diag} & \text{diag} & \\ \hline \end{array} \right) - \frac{1}{2}T\left(\begin{array}{|c|c|c|} \hline & \text{diag} & \\ \hline \text{diag} & \text{diag} & \\ \hline \text{diag} & \text{diag} & \\ \hline \end{array} \right), \\
|79\rangle &= \frac{1}{2}T\left(\begin{array}{|c|c|c|} \hline & \text{diag} & \\ \hline \text{diag} & & \text{diag} \\ \hline \text{diag} & & \text{diag} \\ \hline \end{array} \right) - \frac{1}{2}T\left(\begin{array}{|c|c|c|} \hline \text{diag} & & \\ \hline \text{diag} & & \\ \hline \text{diag} & & \\ \hline \end{array} \right),
\end{aligned}$$

FIG. 17: Base wave vectors $|74\rangle - |79\rangle$.

$$|80\rangle = \frac{1}{4}T\left(\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \blacksquare & \square & \blacksquare \\ \hline \square & \square & \square \\ \hline \end{array}\right) - \frac{1}{4}T\left(\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \blacksquare & \blacksquare & \square \\ \hline \square & \blacksquare & \square \\ \hline \end{array}\right),$$

$$|81\rangle = \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline \square & \square & \blacksquare \\ \hline \square & \square & \square \\ \hline \blacksquare & \square & \square \\ \hline \end{array}\right) - \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \blacksquare \\ \hline \blacksquare & \square & \square \\ \hline \end{array}\right),$$

$$|82\rangle = T\left(\begin{array}{|c|c|c|}\hline \square & \square & \blacksquare \\ \hline \square & \square & \blacksquare \\ \hline \blacksquare & \square & \square \\ \hline \end{array}\right),$$

$$|83\rangle = T\left(\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \blacksquare & \square \\ \hline \blacksquare & \square & \square \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \blacksquare & \square \\ \hline \blacksquare & \square & \square \\ \hline \end{array}\right),$$

$$|84\rangle = T\left(\begin{array}{|c|c|c|}\hline \blacksquare & \square & \square \\ \hline \blacksquare & \square & \square \\ \hline \square & \square & \blacksquare \\ \hline \end{array}\right) - T\left(\begin{array}{|c|c|c|}\hline \blacksquare & \blacksquare & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \blacksquare \\ \hline \end{array}\right),$$

$$|85\rangle = \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \blacksquare & \square & \blacksquare \\ \hline \blacksquare & \square & \blacksquare \\ \hline \end{array}\right) - \frac{1}{2}T\left(\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \blacksquare \\ \hline \square & \square & \blacksquare \\ \hline \end{array}\right).$$

FIG. 18: Base wave vectors $|80\rangle - |85\rangle$.